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Nonlinear H_∞ Method for Control of Wing Rock Motions

Shyh-Pyng Shue* and Ramesh K. Agarwal†
Wichita State University, Wichita, Kansas 67260-0093
and
Peng Shi‡
University of South Australia, The Levels, South Australia 5095, Australia

Control of the nonlinear wing rock motion of slender delta wings using a nonlinear H_∞ robust method is presented. The wing rock motion is mathematically described by a nonlinear, ordinary differential equation with coefficients varying with angle of attack. In the time domain approach, the nonlinear H_∞ robust control problem with state feedback is cast in terms of a Hamilton–Jacobi–Bellman inequality (HJBI). Assuming that the coefficients in the nonlinear equation of the wing rock motion satisfy a norm-bounded nonlinear criterion, the HJBI can be written in a matrix form. The state vector is represented as a series of closed-loop Lyapunov functions that result in reducing the HJBI to an algebraic Riccati inequality along with several other algebraic inequalities. These inequalities can be successively solved to a desired power in the series representation of the state vector in the HJB equation. The results of the nonlinear H_∞ state feedback control are compared with those obtained with the linear H_∞ state feedback control, indicating the necessity of employing nonlinear feedback control for nonlinear dynamics.

I. Introduction

At a high angle of attack, the flowfield of a slender delta wing is characterized by a strongly organized vortical flow emanating from its leading edges. The vortex sheets emanating from the leading edges roll into a pair of vortices. As the angle of attack increases, these vortices interact with each other and the wing to produce a sustained, large-amplitude, rigid-body oscillation known as wing rock.[1] Such oscillations lead to a significant loss in lift and can cause a serious safety problem during maneuvers such as landing or takeoff. The maneuvering envelope of an aircraft exhibiting this behavior is also seriously restricted because the maximum incidence angle is often limited by the onset of wing rock before the occurrence of stall. Therefore, the control of wing rock motion is of significant importance.

Wing rock motion has been mathematically modeled using wind-tunnel data in Refs. 2 and 3. It has been shown that the aerodynamic coefficients of the aircraft change substantially when wing rock motion occurs. If the changes in the aerodynamic coefficients are assumed to be a system uncertainty that needs to be recovered in the control design, then the H_∞ technique can be effectively employed to achieve this goal.

Application of H_∞ control to stochastic and uncertain systems has been extensively studied during the past decade. The important contributions to H_∞ control theory and its application have been made where linear uncertain systems[4,5] are concerned. Zames[6] was the first to introduce the H_∞ concept of control to stochastic systems, followed by important contributions to the development of H_∞ control theory by Francis and Doyle,[7] Glover and Doyle,[8] Petersen,[9] Bernstein and Haddad,[10] and Francis.[11] Petersen[12] was the first to consider the linear H_∞ suboptimal control by state feedback. A pioneering contribution to H_∞ suboptimal control via output feedback has been made by Doyle et al.[13] References 14 and 15 describe several applications based on the formulations of Doyle et al.[13] The work of Doyle et al.[13] has led to the formulation of the problem in terms of finding the optimal value γ of the transfer function. Recently, the mixed H_2/H_∞ control formulations have been developed and are described in Refs. 16 and 17. The application of the H_∞ reduced-order design is given in Ref. 18. The H_∞ estimation (filter) problem is described in Ref. 19. The discrete H_∞ suboptimal control and filter problem has been extensively studied in Ref. 20. However, all of the preceding references are restricted to linear systems.

The early pioneering papers on the application of H_∞ control to nonlinear systems are due to Isidori and Astolfi,[21,22] Isidori,[23,24] and van der Schaft.[25–27] Additional related research in this area can be found in Refs. 28 and 29. These papers show that the nonlinear H_∞ control problems can be cast in terms of a Hamilton–Jacobi–Bellman (HJB) inequality (HJBI) when using full state feedback control and in terms of two HJB inequalities with estimated feedback control. One of the earlier methods of solving the nonlinear optimal control problems based on the HJB equation is the method of regular perturbations, also called the ε-parameter method. Important contributions based on this method can be found in Refs. 30–34. This method assumes that the closed-loop Lyapunov function (CLLF) is an ε-parameter series expansion. The solution of the HJB equation with this description of CLLF gives both the CLLF and the optimal control. The nonlinear optimal control problem is very closely related to the nonlinear H_∞ control because both problems can be cast in terms of the HJBI. Therefore, several concepts from optimal control theory can be employed to determine the relationship between the closed-loop Lyapunov derivative and the smallest value of γ for nonlinear H_∞ control.

In this paper, a nonlinear H_∞ technique is employed to control the nonlinear wing rock motion and to recover the uncertain coefficients in the equation of motion. The CLLF is assumed to be a series based on the powers of the state vector as shown in Refs. 30 and 35. It is shown that a series solution for the state vector and control is obtained by substituting this CLLF into the HJBI. The suboptimal H_∞ series solution is compared with the linear H_∞ suboptimal control for the wing rock motion. It is shown that the nonlinear H_∞ controller can be effectively employed to stabilize all wing rock motions in the domain of interest.

II. Problem Statement and Objective

A. Problem Statement

The nonlinear wing rock equation of motion for an 80-deg slender delta wing can be written as (see Ref. 1)

$$\ddot{\varphi} + \omega^2\varphi = \mu_1\dot{\varphi} + b_1\dot{\varphi}^3 + \mu_2\varphi^2\dot{\varphi} + b_2\varphi\dot{\varphi}^2 + u \qquad (1)$$

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*Research Associate, National Institute for Aviation Research.
†Bloomfield Distinguished Professor and Executive Director, National Institute for Aviation Research. Fellow AIAA.
‡Lecturer, Centre for Industrial and Applicable Mathematics, School of Mathematics.

Table 1 Coefficients for the wing rock motion from Ref. 1

α	c_1	c_2	a_1	a_2	a_3	a_4	a_5
21.5	0.354	0.001	−0.04207	0.01456	0.04714	−0.18583	0.24234
22.5	0.354	0.001	−0.04681	0.01966	0.05671	−0.22691	0.59065
25	0.354	0.001	−0.05686	0.03254	0.07334	−0.3597	1.4681

where φ, $\dot{\varphi}$, and u represent the bank angle, the roll rate, and the imposed control input, respectively. The control input here is the angular acceleration (deg/s^2) that can be generated by the aileron deflection angle. Reference 1 gives the following relations for the coefficients employed in Eq. (1):

$$\omega^2 = -c_1 a_1 \qquad \mu_1 = c_1 a_1 - c_2 \qquad b_1 = c_1 a_3$$

$$\mu_2 = c_1 a_4 \qquad b_2 = c_1 a_5 \tag{2}$$

The values of these coefficients for various angles of attack are shown in Table 1. Here, the values of the coefficients for $\alpha = 22.5$ deg are selected to describe the nominal wing rock model. The differences in the coefficients of the nominal model and those for the two neighboring angles of attack shown in Table 1 are assumed to be the nonlinear parameter uncertainties. By substituting the values of Table 1 into Eq. (2), and by defining the state vector $x = (x_1, x_2)^T = (\varphi, \dot{\varphi})^T$, Eq. (1) can be written in the following state-space form:

$$\dot{x} = A_1 x + A_2(x) + A_3(x) + \Delta A(x) + B_0 u + G_0 w$$

$$y = (C + \Delta C)x \tag{3}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.0201 & 0.0105 \end{bmatrix} \tag{4}$$

$$A_2(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

$$A_3(x) = \begin{bmatrix} 0 \\ 0.02596x_2^3 - 0.1273x_1^2 x_2 + 0.51971 x_1 x_2^2 \end{bmatrix} \tag{6}$$

$$B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{7}$$

and

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8}$$

In Eq. (3), $\Delta A(x)$ is the nonlinear uncertainty in the system, and $G_0 w$ is the disturbance. Note that the higher-power terms in the control matrix and the output matrix are zero in this example. In Ref. 36, it is shown that the disturbance in the lateral motion of the aircraft is in roll acceleration $\ddot{\varphi}$ only. A disturbance in the roll rate acceleration $\ddot{\varphi}$ is, therefore, assumed. Now, introducing a small disturbance w in the state variable $\ddot{\varphi}$ results in

$$\dot{x_2} = \dot{x_2} + w \tag{9}$$

Neglecting the higher-power terms in the description of the gust, the nonlinear state equation for the gust variable from Ref. 36 can, therefore, be described as follows:

$$G_0 = \begin{bmatrix} 0 \\ 1.0 \end{bmatrix} \tag{10}$$

which indicates that the disturbance is generated in the roll rate. The gust effect on the bank angle is indirectly obtained from the roll rate because the roll rate is one time derivative higher than the bank angle. Thus, the state-space model of the wing rock motion is completely established.

B. Objective

The objective is to find a nonlinear robust feedback controller such that the nonlinear uncertain wing rock system (3) is stabilized. To solve this problem, the nonlinear uncertainty of the system is assumed to be bounded as

$$\|\Delta A(x)\| \in \Omega(x) \tag{11}$$

where

$$\Omega(x) = \Delta A_0 \cdot x + \Delta A_1(x) \cdot x + \Delta A_2(x) \cdot x \tag{12}$$

The following constraint is applied to the uncertainties in Eq. (12):

$$\begin{bmatrix} \Delta A_0 \\ \Delta A_1(x) \\ \Delta A_2(x) \end{bmatrix} = \begin{bmatrix} E_0 \\ E_1 \\ E_2 \end{bmatrix} F(x,t)[J_0 \quad J_1(x) \quad J_2(x)] \tag{13}$$

where (E_0, E_1, E_2 and J_0, J_1, J_2) are known, properly dimensioned matrices in which E_0 and J_0 are constant and E_1, E_2 and J_1, J_2 are given as functions of x. Note that F is an unknown time-varying matrix satisfying

$$\|F(x,t)\| \le I \quad \forall t \in [0, t_f] \tag{14}$$

with the elements of F being Lebesgue measurable. Note that the subscripts of the matrices in Eq. (13) represent the power of the state variable. The matrix F is included in the uncertain parameter in the state matrix of the system given by Eq. (3) and is allowed to be state dependent as long as Eq. (14) is satisfied for all possible state trajectories. The matrices E_0, E_1, E_2 and J_0, J_1, J_2 specify how the uncertain parameters in F affect the normal matrices of system (3). It can be shown that the upper bound for F does not cause any loss of generality. Also, F is normalized to satisfy Eq. (14) by properly choosing the matrices E_0, E_1, E_2 and J_0, J_1, J_2.

The following matrix inequality[16] is employed to find the suitable linear formulation for the system satisfying the constraint (13):

$$J_1 F E_1 + E_1^T F^T J_1^T \le (1/\varepsilon^2) J_1 J_1^T + \varepsilon^2 E_1^T E_1 \tag{15}$$

where ε is an arbitrary positive scalar.

III. Method of Analysis

A. Theoretical Formulation

Consider a time-invariant nonlinear system as follows:

$$\dot{x} = A(x) + Bu + Gw \qquad x(0) = x_0$$

$$y = C(x) \qquad A(0) = 0 \qquad C(0) = 0 \tag{16}$$

where $x \in R^n$, $y \in R^p$, $u \in R^m$, and $w \in R^d$. In Eq. (16), x is the state vector, u is the control vector, and w is the disturbance vector. Note that $A(x)$, B, G, and $C(x)$ have compatible dimensions. The objective of the H_∞ control design is to find a robust control u that minimizes the performance index J and the disturbance w that maximizes J:

$$J = \frac{1}{2}\int_0^\infty [A^T(x)A(x) + u^T u - \gamma^2 w^T w] \tag{17}$$

The feedback solution of this min–max problem is

$$u(x) = -B^T V_x^T \tag{18}$$

and

$$w(x) = (1/\gamma^2) G^T V_x^T \tag{19}$$

where

$$H = V_x A(x) + \tfrac{1}{2}C^T(x)C(x) + (1/2\gamma^2)V_x GG^T V_x^T$$
$$-\tfrac{1}{2}V_x BB^T V_x^T \le 0 \tag{20}$$

is the Hamiltonian and $V_x = \partial V(x)/\partial x$ is the derivative of the CLLF $V(x)$. The equality in Eq. (20) is employed to generate the necessary formulations for the nonlinear H_∞ state feedback problem. Substituting Eqs. (18) and (19) into the system (16), the closed-loop nonlinear system becomes

$$\dot{x} = A(x) - [BB^T - (1/\gamma^2)GG^T]V_x^T \tag{21}$$

References 22–26 show that the HJB equation (20) can be reduced to an algebraic Riccati equation if the linearized model is employed. In this paper, the formulations for both the nonlinear state feedback and linearized feedback for the H_∞ control are compared and discussed. Based on the formulation of the nonlinear state feedback, the constraint for the closed-loop linearized state matrix is developed. It is shown that the optimal value of γ in the linear case cannot be used for the nonlinear system because the determinant of the closed-loop linearized state matrix becomes very small. The spatially varying coefficients in the nonlinear state equation make the system diverge immediately.

B. Assumptions and Simplifications

For solving the nonlinear HJB equation (20), we assume that the state and output matrices in the open-loop system can be expressed as

$$A(x) = A_1 x + \sum_{i=2}^{N} A_i(x) \tag{22}$$

and

$$C(x) = C_1 x + \sum_{l=2}^{M} C_l(x) \tag{23}$$

In Eqs. (22) and (23), i and l denote the state matrices with the ith and lth powers of the state vector, respectively. Furthermore, we assume that the CLLF $V(x)$ can be written in a series form as follows:

$$V(x) = \frac{1}{2}x^T X_0 x + \sum_{i=3}^{N+1} V^i(x) \qquad \text{if} \qquad N \text{ is odd}$$
$$V(x) = \frac{1}{2}x^T X_0 x + \sum_{i=3}^{N+2} V^i(x) \qquad \text{if} \qquad N \text{ is even} \tag{24}$$

Now, the derivative of CLLF can be written as

$$V_x(x) = x^T X_0 + \sum_{i=3}^{N+1} V_x^i(x) \tag{25}$$

We further assume that

$$V_x^{i^T}(x) = \hat{X}_{i-1}(x)\cdot x \tag{26}$$

$$A_i(x) = \hat{A}_{i-1}(x)\cdot x \tag{27}$$

and

$$C_l(x) = \hat{C}_{l-1}(x)\cdot x \tag{28}$$

Note that Eqs. (26–28) are based on the assumptions stated in Refs. 30 and 35. Also, $\hat{X}_i$ and $\hat{A}_i$ are $n\times n$ matrices, and $\hat{C}_i$ is a $p\times n$ matrix. Assumptions (26–28) allow system (16) to be written in the following form:

$$\dot{x} = \begin{bmatrix} \hat{A}_0 \\ \hat{A}_1(x) \\ \vdots \\ \hat{A}_{N-1}(x) \end{bmatrix}^T \begin{bmatrix} x \\ x \\ \vdots \\ x \end{bmatrix} + Bu + Gw \tag{29}$$

$$y = \begin{bmatrix} \hat{C}_0 \\ \hat{C}_1(x) \\ \vdots \\ \hat{C}_{M-1}(x) \end{bmatrix}^T \begin{bmatrix} x \\ x \\ \vdots \\ x \end{bmatrix} \tag{30}$$

The derivative of the CLLF can be written as

$$V_x^T(x) = \begin{bmatrix} \hat{X}_0 \\ \hat{X}_1(x) \\ \vdots \\ \hat{X}_{N-1}(x) \end{bmatrix}^T \begin{bmatrix} x \\ x \\ \vdots \\ x \end{bmatrix} \tag{31}$$

where $\hat{X}_0, \ldots, \hat{X}_{N-1}(x)$ are symmetric matrices. Also, $A_1 = \hat{A}_0$ and $\hat{C}_0 = C_1$.

The advantage of writing the state equation in a vector form is that it allows the HJB equation (18) to be expressed in a matrix form. This facilitates the series solution of the HJB equation to a desired power of the state vector. We define $\hat{A}_1$ as a null matrix, and

$$\hat{A}_2 = \begin{bmatrix} 0 & 0 \\ -0.1273x_1x_2 & 0.02596x_2^2 + 0.5197x_1x_2 \end{bmatrix} \tag{32}$$

C. Methodology

The HJB equation (20) was formulated for a system without uncertain coefficients. For the nonlinear wing rock system there are uncertain coefficients that are assumed to satisfy the constraints given by Eqs. (13–15). The following procedure is employed to find the HJB equation for a system with uncertain coefficients.

Because the HJB equation is a scalar and the Lyapunov function is assumed to be symmetric, the HJB equation (20) can be modified for an uncertain system as follows:

$$H = (H + H^T)/2 = \tfrac{1}{2}\{V_x[A(x) + \Delta A(x)]$$
$$+ [A(x) + \Delta A(x)]^T V_x^T + C^T(x)C(x)$$
$$-V_x[B(x)B^T(x) - \gamma^{-2}G(x)G^T(x)]V_x^T\} \le 0 \tag{33}$$

Arranging Eq. (33) based on the powers of the state vector results in the following matrix form:

$$H = \frac{1}{2}\begin{bmatrix} x \\ x \\ \vdots \\ x \end{bmatrix}^T \begin{bmatrix} H_{11} & H_{12} & H_{13} & \cdots \\ H_{12}^T & H_{22} & \ddots & \vdots \\ H_{13}^T & \ddots & \ddots & \ddots \\ \vdots & \cdots & \ddots & H_{ii} \end{bmatrix} \begin{bmatrix} x \\ x \\ \vdots \\ x \end{bmatrix} = 0 \tag{34}$$

where

$$H_2 = H_{11} = \hat{X}_0(\hat{A}_0 + \Delta A_0) + (\hat{A}_0 + \Delta A_0)^T\hat{X}_0$$
$$- \hat{X}_0[BB^T - (1/\gamma^2)GG^T]\hat{X}_0 + \hat{C}_0^T C_0 = 0 \tag{35}$$

$$H_3 = H_{12} + H_{12}^T = \hat{X}_0(\hat{A}_1 + \Delta A_1) + (\hat{A}_1 + \Delta A_1)^T\hat{X}_0$$
$$+ \hat{X}_1(\hat{A}_0 + \Delta A_0) + (\hat{A}_0 + \Delta A_0)\hat{X}_1 + \hat{C}_1^T C_0 + C_0^T C_1$$
$$- \hat{X}_1[BB^T - \gamma^{-2}GG^T]\hat{X}_0 - \hat{X}_0[BB^T - \gamma^{-2}GG^T]X_1^T = 0 \tag{36}$$

$$H_4 = H_{13} + H_{13}^T + H_{22} = \hat{X}_0(\hat{A}_2 + \Delta A_2) + (\hat{A}_2 + \Delta A_2)^T \hat{X}_0$$
$$+ \hat{X}_1(\hat{A}_1 + \Delta A_1) + (\hat{A}_1 + \Delta A_1)^T \hat{X}_1 + \hat{X}_2(\hat{A}_0 + \Delta A_0)$$
$$+ (\hat{A}_0 + \Delta A_0)^T \hat{X}_2 - \hat{X}_2[BB^T - \gamma^{-2}GG^T]\hat{X}_0$$
$$- \hat{X}_0[BB^T - \gamma^{-2}GG^T]\hat{X}_2 + C_0^T C_2 + C_1^T C_1 + C_2^T C_0 = 0 \tag{37}$$

$$H_5 = H_{14} + H_{14}^T + H_{23} + H_{23}^T = \sum_{i=0}^{L=5-2} \big[\hat{X}_i(\hat{A}_{L-i} + \Delta A_{L-i})$$
$$+ (\hat{A}_{L-i} + \Delta A_{L-i})^T \hat{X}_i\big] - \sum_{j\ge 0,\ r\ge 0,\ \text{and}\ j+r=L}^{L=5-2} \hat{X}_j[BB^T$$
$$- \gamma^{-2}GG^T]X_r + \sum_{i=0}^{L=5-2} \big[C_i^T C_{L-i}\big] = 0 \tag{38}$$

and so on. Note that the reason for having the highest power of the HJB matrix coefficients equal to $L = N - 2$ in Eq. (38) is that each matrix coefficient has a state vector on both sides. Although the power of the coefficients for the HJB equation is N, the power of the interpolated coefficients in the HJB matrix is $L = N - 2$. Rearranging the HJB equation (34) based on the powers of the state vector gives $H_2 = 0$, $H_3 = 0$, $H_4 = 0, \ldots$. Thus, the following equations for the series solution are obtained.

1. *Second-Power Term of the HJB Equation*

The second-power term in the HJB equation represents the linearized model of the nonlinear H_∞ problem with uncertainty. Applying the constraint given by Eq. (15) results in the second-power term of the HJB equation changing into an algebraic Riccati equation (ARE):

$$\hat{X}_0\hat{A}_0 + \hat{A}_0^T\hat{X}_0 + \varepsilon^2 E_0^T E_0 - \hat{X}_0\big[BB^T - \gamma^{-2}GG^T$$
$$- \varepsilon^{-2} J_0 J_0^T\big]\hat{X}_0 + \hat{C}_0^T C_0 = 0 \tag{39}$$

The preceding ARE has a symmetric, positive, semidefinite solution if and only if the following conditions are satisfied: 1) the pair $(\hat{A}_0, B)$ is stabilizable, 2) the triplet $(\hat{A}_0, G, J_0)$ is stabilizable, and 3) the triplet $(\hat{C}_0, E_0, \hat{A}_0)$ is detectable. Therefore, the optimal values of γ and $\hat{X}_0$ can be determined by following the procedure given in Ref. 22, so long as the closed-loop linear state matrix A_c is stable and X_0 is symmetric positive semidefinite.

2. *Third-Power Term of the HJB Equation*

The third-power term of the HJB equation given by Eq. (36) can be simplified as follows:

$$H_3 = H_{12} + H_{12}^T = \hat{A}_c^T\hat{X}_1 + \hat{X}_1\hat{A}_c + Q_1(x) = 0 \tag{40}$$

where

$$Q_1(x) = \hat{X}_0\hat{A}_1 + \hat{A}_1^T\hat{X}_0 - \varepsilon^{-2}(J_1 J_0^T + J_0 J_1^T)$$
$$+ \hat{C}_1 C_0^T + C_0^T C_1 + \varepsilon^2(E_0^T E_1 + E_1^T E_0) \tag{41}$$

Note that A_c is the linear, closed-loop state matrix given by

$$A_c = A - [BB^T - \gamma^{-2}GG^T]\hat{X}_0 \tag{42}$$

In Eq. (41), all coefficients in $Q_1(x)$ are known. In addition, all terms of Eq. (41) are associated with the first power of state variables. Therefore, the third-power term of the HJB equation can be seen as a first power of the closed-loop Lyapunov equation. This equation is linear and, therefore, can be solved using a linear algebraic process.

3. *Fourth-Power Term of the HJB Equation*

The fourth-power term of the HJB equation is given by Eq. (37). As in the preceding case, this equation can be simplified as follows:

$$H_4 = H_{13} + H_{13}^T + H_{22} = \hat{A}_c^T\hat{X}_2 + \hat{X}_2\hat{A}_c + Q_2(x) = 0 \tag{43}$$

where

$$Q_2 x = \hat{X}_0\hat{A}_2 + \hat{A}_2^T\hat{X}_0 + \hat{X}_1\hat{A}_1 + \hat{A}_1^T\hat{X}_1$$
$$- \varepsilon^{-2}(J_0 J_2^T + J_1 J_1^T + J_2 J_0^T) + (\hat{C}_2 C_0^T + C_0^T C_2 + C_1^T C_1)$$
$$+ \varepsilon^2(E_0^T E_2 + E_1^T E_1 + E_2^T E_0) \tag{44}$$

Equation (44) can be written in a compact form:

$$Q_2(x) = Q_L(x) = \sum_{i=0}^{S=L-1} \big[\hat{X}_i\hat{A}_{S-i+1} + \hat{A}_{S-i+1}^T\hat{X}_i\big]$$
$$- \varepsilon^{-2}\sum_{i=0}^{L}\big[J_i J_S^T\big] + \sum_{i=0}^{L}\big[C_i^T C_{L-i}\big] + \varepsilon^2\sum_{i=0}^{L}\big[E_i^T E_{L-i}\big] \tag{45}$$

with $L = 2$. Note that i, j, K, r, and S are positive integers. Equation (44) represents the second-power expansion of the closed-loop Lyapunov equation. Note that the fourth-power term in the HJB equation, namely, H_4, is solved to obtain the second-power Lyapunov coefficient $\hat{X}_2(x)$ that contains the second-power of the state vector. If the HJB equation is successively solved from zero power to first power of the state vector, the parameters inside $Q_2(x)$ will all be known. Again, the procedure to solve $\hat{X}_2(x)$ involves an algebraic process only.

4. *Higher-Power Terms in the HJB Equation*

Following the case in Sec. III.C.3, the solution of higher-power terms of the CLLF shown in Eq. (38) can be obtained using the appropriate higher-power HJB equation as follows:

$$H_I = \hat{A}_c^T\hat{X}_{I-2}(x) + \hat{X}_{I-2}(x)\hat{A}_c + Q_{I-2}(x) = 0 \tag{46}$$

where

$$Q_{I-2}(x) = Q_L(x) = \sum_{i=0}^{S=L-1} \big[\hat{X}_i\hat{A}_{S-i+1} + \hat{A}_{S-i+1}^T\hat{X}_i\big]$$
$$- \varepsilon^{-2}\sum_{i=0}^{L}\big[J_i J_S^T\big] + \sum_{i=0}^{L}\big[C_i^T C_{L-i}\big] + \varepsilon^2\sum_{i=0}^{L}\big[E_i^T E_{L-i}\big] \tag{47}$$

Note that $L = I - 2$ and $I \ge 3$. The preceding formulations allow the nonlinear CLLF to be computed as a series based on the powers of the state vector. Note that as long as the system is solved successively, the terms on the right hand side of Eq. (47) are all known. This allows the series solution of the closed-loop Lyapunov equation to be obtained by algebraic processes only. In fact, the algebraic Riccati equation (ARE) (39) can also be written in an analogous manner as

$$\hat{X}_0\hat{A}_c + \hat{A}_c^T\hat{X}_0 + Q_0 = 0 \tag{48}$$

where

$$Q_0 = \hat{X}_0[BB^T - (1/\gamma^2)GG^T]\hat{X}_0 + \hat{C}_0^T C_0 \tag{49}$$

The nonlinear feedback controller and the exogenous input are obtained as

$$u = -B^T(x)[\hat{X}_0 \quad \hat{X}_1 \quad \cdots \quad \hat{X}_{N-1}][x \quad x \quad \cdots \quad x]^T \tag{50}$$

and

$$w = (1/\gamma^2)G^T(x)[\hat{X}_0 \quad \hat{X}_1 \quad \cdots \quad \hat{X}_{N-1}][x \quad x \quad \cdots \quad x]^T \tag{51}$$

In the next section, stability issues related to the H_∞ nonlinear feedback control of nonlinear systems and the stability of the nonlinear CLLF are discussed.

D. Stability of the CLLF

For H_∞ control of linear systems, the optimal value of γ is usually selected such that some of the closed-loop eigenvalues are very close to the imaginary axis. As long as the solution of the ARE $\hat{X}_0$ is positive definite and the real parts of all of the eigenvalues of A_c are located in the left complex plane, this value of γ can be considered the optimal value. Because some closed-loop eigenvalues are very close to the imaginary axis, the determinant of the closed-loop state matrix is very small. In the nonlinear H_∞ feedback control of nonlinear systems, all coefficients of the CLLF are determined by the closed-loop Lyapunov equations. These equations are directly formulated in terms of the linear, closed-loop state matrix A_c. Because the determinant of A_c is very small, the coefficients of the CLLF become very large. These coefficients also depend on $Q_1(x), Q_2(x), \ldots, Q_i(x)$. The higher power of $Q_i(x)$ involves $Q_{i-1}(x)$ and lower powers. The values of $Q_i(x)$ increase at each successive value of the ith step because of the cumulative effect of a small value of A_c. This situation has a strongly adverse effect on the stability of the system. Thus, the optimal value of γ obtained from linear analysis cannot be used for the nonlinear H_∞ feedback control of nonlinear systems. The suitable value of γ for the H_∞ nonlinear feedback problems is discussed later.

We employ the Hessian matrix of the CLLF to determine the suitable value of γ for the nonlinear H_∞ problem. The Hessian matrix of the CLLF is given by

$$M_H = \frac{\mathrm{d}^2}{\mathrm{d}x^2} V(x) \tag{52}$$

Note that the Hessian matrix of a scalar function is equivalent to the Jacobian matrix of a vector function. According to the second method of Lyapunov,[37] for the equilibrium state $x_e = 0$ of the system $\dot{x} = X(x)$ to be asymptotically stable in the global sense, it is sufficient that there exists a scalar, continuous Lyapunov function $V(x)$ satisfying the following conditions: 1) $V(0) = 0$, 2) $V(x) > 0$ for all $x \neq 0$, and 3) $\dot{V}(x) = (\partial V/\partial x)\dot{x} < 0$ for all $x \neq 0$.

If the CLLF is positive definite and the HJB equation is satisfied, then the controlled system will be asymptotically stable in the global sense as shown here

$$\begin{aligned} V(x) &= (\partial V/\partial x)\dot{x} = V_x(A(x) + B(x)u + G(x)w) \\ &\le -\left\{ \tfrac{1}{2}C^T(x)C(x) + \tfrac{1}{2}V_x(x)[B(x)B^T(x) \right. \\ &\quad \left. - (1/\gamma^2)G(x)G^T(x)]V_x^T(x) \right\} \le 0 \end{aligned} \tag{53}$$

The proof of the above inequality (53) is given in Ref. 27.

The positive definiteness of the CLLF is determined by the Hessian matrix of the CLLF. This Hessian matrix can be expressed as follows:

$$M_H = \hat{X}_0 + \frac{\mathrm{d}}{\mathrm{d}x}[\hat{X}_1 \cdot x] + \frac{\mathrm{d}}{\mathrm{d}x}[\hat{X}_2 \cdot x] + \cdots \tag{54}$$

For all $x = 0$, the preceding matrix reduces to the solution of the ARE. This matrix is positive definite if the conditions for the ARE are satisfied. This guarantees that the controlled system is stable in the small area around the origin. Because the controlled system satisfies the HJB equation, the condition for the CLLF that $\dot{V}(x) \le 0$ is automatically satisfied. If the CLLF can be found to be positive for $x \neq 0$, then the controlled system will be asymptotically stable in the global sense. Note that Eq. (54) is expressed based on the powers of the state vector. To enlarge the stability of the CLLF, the following conditions are required:

$$\hat{X}_0 > 0 \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\hat{X}_2 \cdot x] > 0 \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\hat{X}_4 \cdot x] > 0, \ldots \tag{55}$$

Note that the positive definiteness of the Hessian matrix is determined by the even-power solution of the HJB equation. Rewriting Eq. (39) in matrix form gives

$$M_H = \begin{bmatrix} I_{n\times n} \\ I_{n\times n} \\ \vdots \\ \vdots \end{bmatrix}^T \begin{bmatrix} \hat{X}_0 & \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\hat{X}_1 \cdot x\right) & \cdots \\ \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\hat{X}_1 \cdot x\right)^T & \frac{\mathrm{d}}{\mathrm{d}x}(\hat{X}_2 \cdot x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} I_{n\times n} \\ I_{n\times n} \\ \vdots \\ \vdots \end{bmatrix} \tag{56}$$

where $I_{n\times n}$ is an $n \times n$ identity matrix. As long as $\hat{X}_0$ is positive definite and

$$\begin{bmatrix} \hat{X}_0 & \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}x}(\hat{X}_1 \cdot x)\right] \\ \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}x}(\hat{X}_1 \cdot x)\right]^T & \frac{\mathrm{d}}{\mathrm{d}x}(\hat{X}_2 \cdot x) \end{bmatrix} \ge 0 \tag{57}$$

the nonlinear controlled system will have larger stable region than the linearized controller. The higher the order of the positive-definite interpolated matrix of the Hessian matrix in Eq. (56), the larger will be the region of stability for the controlled system for all $x = 0$. As long as the Hessian of the CLLF is positive definite, the system will be stable in the global sense. This is because the controlled system satisfies the second method of Lyapunov and the convex theory of the calculus of variations.[37] However, even if the global solution cannot be found by this procedure, this method will always provide a larger stable region for the nonlinear controlled system than that obtained with the linearized controller.

IV. Results and Discussion

A. Solution of CLLF

To find the CLLF derivative with respect to the state vector for the nonlinear wing rock system, the ARE is solved first. It can be verified easily that Eq. (39) has a minimum value, $\gamma = 1.0206$, using the bisection method. However, the control based on this value causes the determinant of A_c to be very small. To avoid a small value for the determinant of A_c, it is necessary to take the value of γ to be somewhat larger. Therefore, γ is selected to be 1.5. The corresponding solution of the ARE is obtained as

$$P = \hat{X}_0 = \begin{bmatrix} 1.9797 & 1.3818 \\ 1.3818 & 2.7223 \end{bmatrix}$$

The associated linear, closed-loop state matrix is given by

$$A_c = \hat{A}_0 - \left[B_0 B_0^T - (1/\gamma^2)G_0 G_0^T\right]\hat{X}_0 = \begin{bmatrix} 0.0 & 1 \\ -0.7878 & -1.5019 \end{bmatrix}$$

Following the procedure outlined in the preceding section, the following CLLF derivatives with respect to the state vector are determined to be

$$\hat{X}_1(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\hat{X}_2(x) = \begin{bmatrix} -0.05081x_1x_2 + 0.00077x_2^2 & -0.1154x_1x_2^2 \\ -0.1154x_1x_2^2 & 0.0787x_1x_2 + 0.03091x_2^2 \end{bmatrix}$$

The CLLF derivative with respect to the state vector is found to be

$$V_x(x) = \begin{bmatrix} 1.9797x_1 + 1.38186x_2 - 0.11461x_1^2x_2 - 0.208x_1x_2^2 - 0.0508x_2^3 \\ 1.3818x_1 + 2.7223x_2 - 0.1154x_1^2x_2 + 0.0787x_1x_2^2 + 0.0303x_2^3 \end{bmatrix}$$

Therefore, the nonlinear H_∞ controller can be computed as

$$u(x) = -B^T V_x^T = -1.3818x_1 - 2.7223x_2 + 0.1154x_1^2x_2 - 0.0787x_1x_2^2 - 0.0303x_2^3$$

and the worst-case feedback disturbance as

$$w(x) = (1/\gamma^2)G^T V_x^T = 0.6141x_1 + 1.2099x_2 - 0.0513x_1^2x_2 + 0.035x_1x_2^2 + 0.0135x_2^3$$

B. Comparison of Linear and Nonlinear Control Laws

The wing rock motion without any control is shown in Fig. 1. It is a limit cycle vibration. The control architectures for control of wing rock motion using H_∞ linear and nonlinear control laws are employed to show the potential of the proposed method. The linear feedback model is the nonlinear system linearized about the equilibrium point (0, 0). The H_∞ feedback control of this linear model is the same as the second-power CLLF derivative, that is,

$$u = -B^T \hat{X}_0 x = -1.3818x_1 - 2.7223x_2$$

$$w = (1/\gamma^2)G^T \hat{X}_0 x = 0.6141x_1 + 1.2099x_2$$

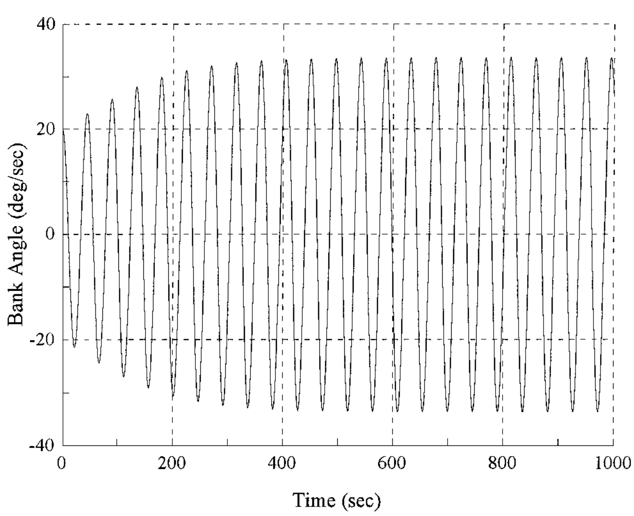

Fig. 1 Wing rock motion.

In Fig. 1, the roll angle of the wing rock motion is within ±40 deg. It is necessary that the developed control law can be used for any initial condition of the roll angle based on this limit of ±40 deg. Therefore, several different initial conditions from small to large are used to compare the stability of the controlled wing rock motions using the linear and nonlinear control laws. These conditions gradually increase the initial value from small to the limit of ±40 deg to evaluate the difference between the linear and nonlinear control laws. Results for the control of wing rock motion employing linear and nonlinear control laws under different initial conditions are shown in Figs. 2–5. Figure 2 shows that the linear H_∞ control and nonlinear H_∞ control are different even when the small initial condition

$$[x_1(0) \quad x_2(0)]^T = [20 \text{ deg} \quad 0]^T$$

is employed. Figure 2 also shows that the linear H_∞ feedback control law cannot stop the wing oscillation in a short time. However, the nonlinear part of the nonlinear H_∞ feedback control makes a big contribution to the closed-loop system that allows the wing oscillation to reach steady state within 6 s. In addition, the linear and nonlinear H_∞ feedback control inputs, as shown in Fig. 3, show the monotonic behavior of the nonlinear H_∞ control in contrast to the linear control. Furthermore, as the initial roll rate becomes larger, for example for

$$[x_1(0) \quad x_2(0)]^T = [20 \text{ deg} \quad 120 \text{ deg/s}]^T$$

the linear-H_∞-controlled system diverges as shown in Fig. 4. However, the nonlinear H_∞ controller still provides an excellent performance for the nonlinear system. Figure 5 shows that the nonlinear system controlled by the nonlinear H_∞ feedback remains convergent even for very large initial conditions as high as

$$[x_1(0) \quad x_2(0)]^T = [100 \text{ deg} \quad 100 \text{ deg/s}]^T$$

These results show that the nonlinear system controlled by the nonlinear H_∞ feedback is stable in a much larger domain than the H_∞

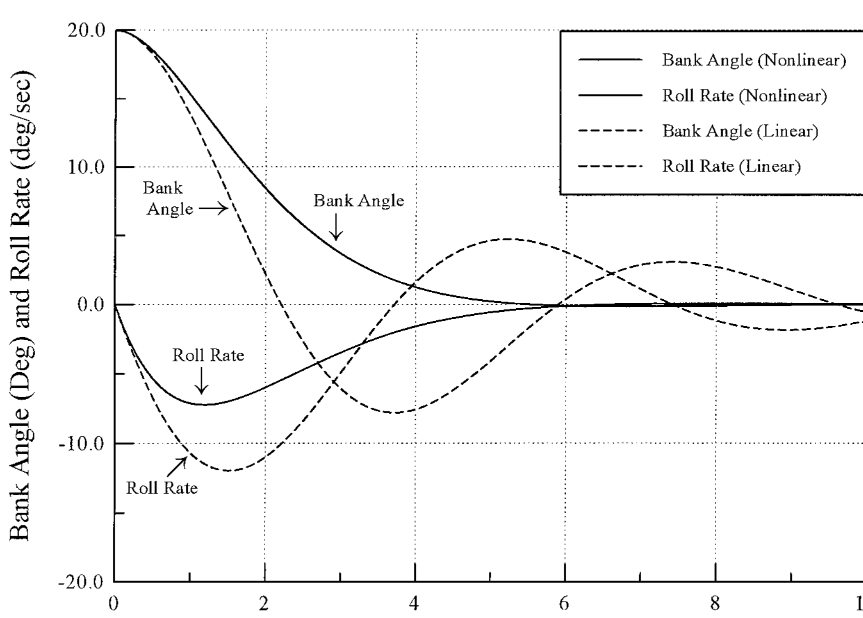

Fig. 2 Responses of the controlled system by linear and nonlinear feedback for small initial conditions.

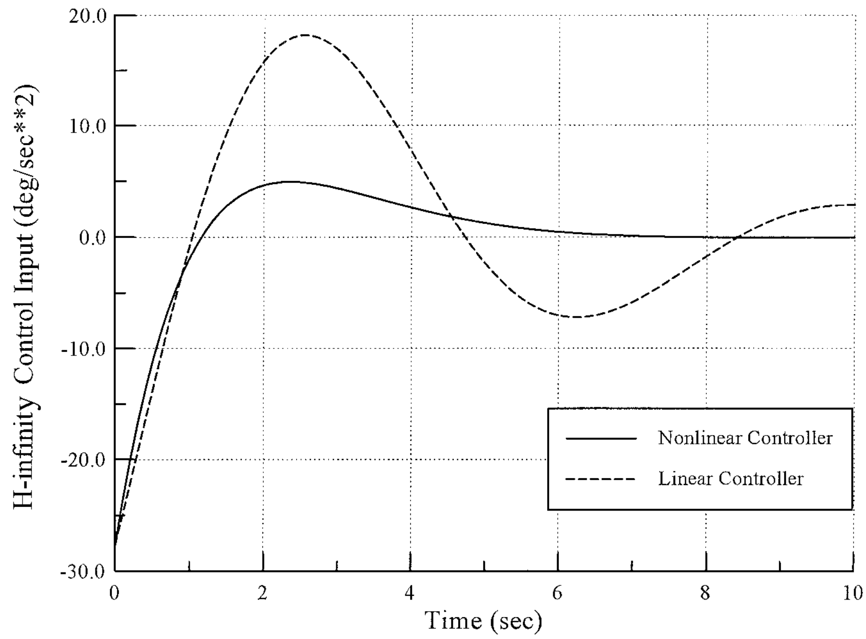

Fig. 3 Time histories of the linear and nonlinear H_∞ feedback controller for the small initial condition.

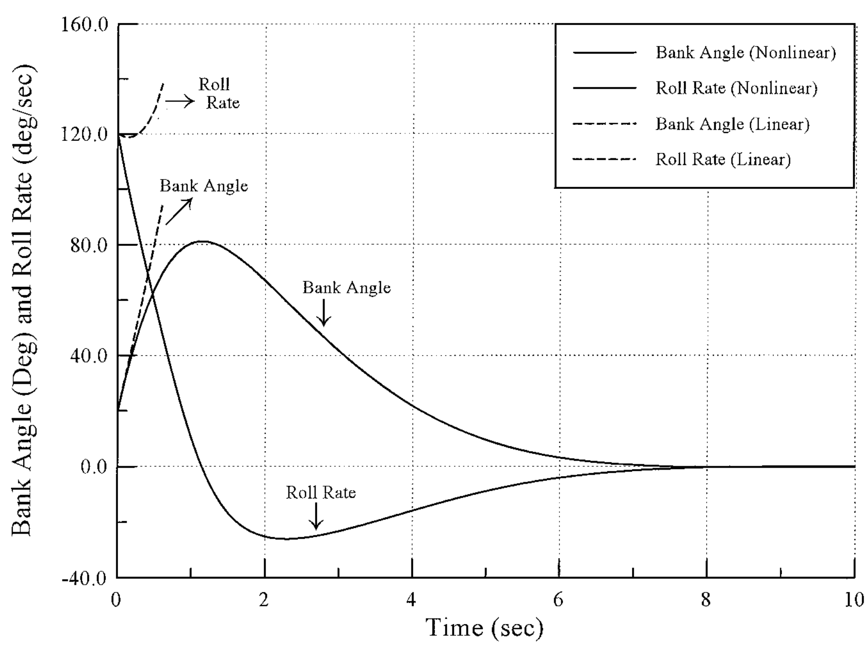

Fig. 4 Responses of the controlled system by linear and nonlinear feedback for large initial conditions.

linear feedback control. Similar results using linear and nonlinear control theory have been obtained by Garrard and Jordan.[38]

C. Discussion on the choice of γ

The nonlinear H_∞ control theory has recently been developed in Refs. 21–29. The present paper applies the nonlinear H_∞ technique to the control of nonlinear wing rock motion. The effect of the optimal value γ on the nonlinear H_∞ problem is discussed. It is shown that the optimal (minimum) value of γ is not suitable for the nonlinear H_∞ controller. For reasons of stability and performance, it is necessary to increase the value of γ for the nonlinear H_∞ control problem. It is also shown that the system controlled by linearized H_∞ feedback may not be stable in a sufficiently large domain regardless of how large the value of γ. Using the nonlinear H_∞ control, the stable region of the controlled system can be significantly extended compared to the linear H_∞ control.

Selection of the appropriate value of γ is very important in the nonlinear H_∞ control problem. Usually, the value of γ is chosen so that the second derivative of the CLLF remains positive definite for all $x \neq 0$. It ensures a larger stable region for the nonlinear H_∞ problem. If the highest power of the state equation is very large, the value of γ can be selected such that the determinant of A_c is larger than unity. The appropriate choice of γ allows the coefficients of the CLLF derivative to be reduced after each calculation. This is because each CLLF derivative is a CLLF that is determined by A_c. By the appropriate selection of γ, the nonlinear, closed-loop system has a larger stable domain around the origin. Whether γ can be selected such that the closed-loop, nonlinear system is asymptotically stable in the global sense requires further study.

In the current wing rock example, the value of γ is selected very close to the optimal value. The closed-loop, nonlinear system using nonlinear H_∞ control has a larger stable region than the nonlinear wing rock system controlled by the linearized H_∞ control.

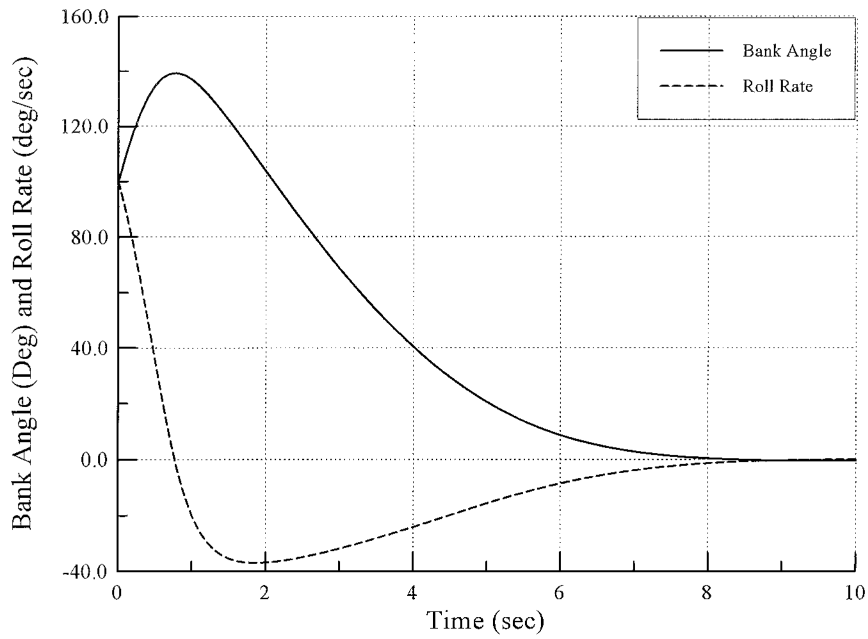

Fig. 5 Responses of the nonlinear feedback controlled system for large initial conditions.

V. Conclusions

A set of closed-loop Lyapunov equations have been developed for calculating the CLLF derivative for the nonlinear H_∞ control problem. It has been shown that the nonlinear H_∞ problem is governed by the HJB inequality. This HJB inequality is reduced to an algebraic Riccati inequality, along with several other algebraic inequalities, by expanding the derivative of the CLLF in a series form of the state vector. It is shown that these algebraic equations can be solved successively based on the state variable power of the HJB inequality. A nonlinear wing rock equation of motion is employed as an application example. It is shown that the nonlinear wing rock system with uncertainties can be effectively controlled by the nonlinear H_∞ feedback, which provides a larger stable region than the linearized H_∞ control.

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